*Linear Mixed Equations and their Analytic Solutions.**

By R. D. Carmichael.

The most important early papers on mixed equations are those by Biot† and Poisson.‡ Each of them deals with the practical integration of special equations and the application of the results to the solution of geometrical problems. The principal contents of both papers, together with references to the previous literature, are to be found in Lacroix's *Traité du Calcul.* §

Questions of an analytic nature have been further treated by Cesaro, Lecornu, Brajtzew and others. ||

Several applications of mixed equations in the treatment of geometrical problems are to be found in memoirs by Puiseux and Combescure.¶

From this brief bibliography it appears that the general questions as to the existence and the nature of solutions of mixed equations have not been treated. It is the object of this paper to consider these questions for the system of mixed equations,

$$D_i f_i(x) = \sum_{j=1}^{n} \psi_{ij}(x) f_j(x), \qquad i = 1 \ldots . n,$$

where

$$\psi_{ij}(x) = \psi''_{ij} x^{-2} + \psi'''_{ij} x^{-3} + \ldots ., \qquad |x| \geqq R;$$

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* Read before the American Mathematical Society, October 29, 1910.

† Biot, "Sur les Équations aux Différences Mêlées," *Mémoires de l'Institut de France, Savans étrangers*, Vol. I (1806), pp. 296–327.

‡ Poisson, "Sur les Équations aux Différences Mêlées," *Journal de l'École Polytechnique*, Cahier XIII (1806), pp. 126–147.

§ Third edition, Vol. III (1819), pp. xix, 575–600.

|| Cesaro, "Sur une Équation aux Différences Mêlées," *Nouvelles Annales de Mathématiques*, Ser. 3, Vol. IV (1885), pp. 36–40; Lecornu, "Sur Certaines Équations aux Différences Mêlées," *Bulletin de la Société Mathématique de France*, Vol. XXVII (1899), pp. 153–160; Brajtzew, "On the Question of the Integration of Linear Mixed Equations by the aid of Definite Integrals" (Russian), *Moscow Mathematical Collection*, Vol. XXII (1901), pp. 275–284. See also the following papers: Gregory, *Cambridge Mathematical Journal*, Vol. I (1839), p. 54; Walton, *Quarterly Journal*, Vol. X (1870), pp. 248–253; Oltramare, *Association Français, Compte rendu* XX (1891), pp. 66–72; Oltramare, *Ibid.*, XXIV (1895), pp. 175–186; Lémeray, *Edinburgh Mathematical Society Proceedings* (1898), pp. 13–44.

¶ Puiseux, "Sur les Développées et les Développantes des Courbes Planes," *Liouville's Journal*, Vol. IX (1844), pp. 377–399; Combescure, "Sur Quelques Questions qui Dépendent des Différences Finies ou Mêlées," *Annales de l'École Normale Supérieure*, Ser. 2, Vol. III (1874), pp. 305–362.

the symbols D_i, $i = 1, \ldots, k$, denote differentiation with respect to x, and the symbols D_i, $i = k+1, \ldots, n$, are identical with the symbol Δ of the difference calculus. It will be noticed that such a system is analogous in form to a system of differential equations of the first order having infinity for an ordinary point.

In Section I two formal expansions, involving arbitrary constants and arbitrary periodic functions are obtained, both of which formally satisfy the given system of mixed equations.

In Section II we demonstrate the convergence of the formal expansions throughout properly determined regions and obtain fundamental existence theorems in two forms, and thus exhibit a remarkable duality in the solutions of mixed equations. A method of applying these results to a single equation of order higher than the first is indicated in a brief remark.

Section III contains an application of the preceding theorems to a system of difference equations. Two fundamental systems of solutions are obtained.

In Section IV we point out an essential difference between mixed equations and either differential or difference equations. This difference is due to the remarkable manner in which the arbitrary elements enter into the solution of the mixed equations.

I.

Let D_i, $i = 1 \ldots n$, denote a set of n operations, the first k of them being each identical with D and the remaining ones with Δ, where k is some one of the numbers $0, 1, 2, \ldots, n$, and where D is the symbol of differentiation and Δ is the symbol of differences, that is, $\Delta f(x) = f(x+1) - f(x)$, and form the system of equations

$$D_i f_i(x) = \sum_{j=1}^{n} \psi_{ij}(x) f_j(x), \qquad i = 1 \ldots n, \tag{1}$$

where

$$\psi_{ij}(x) = \psi_{ij}'' x^{-2} + \psi_{ij}''' x^{-3} + \ldots, \qquad |x| \geqq R. \tag{2}$$

In applying the method of successive approximation to the problem of finding a solution of (1) we write the auxiliary equations

$$\left.\begin{aligned} D_i f_i^{(1)}(x) &= 0, \\ D_i f_i^{(2)}(x) &= \sum_{j=1}^{n} \psi_{ij}(x) f_j^{(1)}(x), \\ D_i f_i^{(3)}(x) &= \sum_{j=1}^{n} \psi_{ij}(x) f_j^{(2)}(x), \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \quad i = 1 \ldots n, \tag{3}$$

and by means of them define a sequence of approximation functions which lead to a set of formal expansions satisfying (1).

Each of the above auxiliary systems (3), after the first, is of the form

$$D_i g_i(x) = \eta_i(x), \qquad i = 1 \ldots . n. \tag{4}$$

Considering first those equations of this system in which D_i denotes Δ we have evidently two particular formal solutions

$$g_i(x) = -\sum_{\nu=0}^{\infty} \eta_i(x+\nu), \tag{5}$$

$$i = k+1 \ldots . n.$$

$$g_i(x) = \sum_{\nu=1}^{\infty} \eta_i(x-\nu), \tag{6}$$

The other equations of system (4), namely those in which D_i denotes D, have each two formal solutions analogous respectively to (5) and to (6). We may conveniently write these in the form

$$g_i(x) = -\int_x^{+\infty} \eta_i(x)\,dx, \tag{7}$$

$$i = 1 \ldots . k,$$

$$g_i(x) = -\int_x^{-\infty} \eta_i(x)\,dx, \tag{8}$$

where in the first case the path of integration extends from x to infinity in a direction parallel to the positive real axis, and in the second case it extends from x to infinity parallel to the negative real axis. Equations (5) and (7) may be taken together as constituting a single formal solution of (4), and this solution we write in the form

$$g_i(x) = S_{xi}^{+}(\eta_i), \qquad i = 1 \ldots . n, \tag{9}$$

where $S_{xi}^{+}(\eta_i)$, for a particular i, is identical with that second member of (5) or of (7) which has the same subscript i. Likewise from (6) and (8) we have a second formal solution of (4), and this we write in the form

$$g_i(x) = S_{xi}^{-}(\eta_i), \qquad i = 1 \ldots . n, \tag{10}$$

where $S_{xi}^{-}(\eta_i)$, for a particular i, is identical with that second member of (6) or of (8) which has the same subscript i. The solution of the first system in the set (3) we shall denote by $t_i(x)$, $i = 1 \ldots . n$. For the equations in which D_i is D it is evident that $t_i(x)$ is a constant; for the other equations it is a periodic function of period 1.

We are now in position to obtain successively a set of particular formal solutions of systems (3); we employ in the first instance the formal solutions (9) of (4). When any particular system of (3) is solved, the set of values so found is substituted in the following system and this in turn is then solved. In each case those particular solutions are selected which will be convenient later on; from each system after the first these solutions are obtained by

operating on the equations throughout by S_{xi}^{+} and then adding $t_i(x)$ to the resulting second member. Thus we have

$$
\begin{aligned}
&f_i^{(1)}(x) = t_i(x),\\
&f_i^{(2)}(x) = t_i(x) + S_{xi}^{+}\big(\sum_{j=1}^{n}\psi_{ij}t_j\big),\\
&f_i^{(3)}(x) = t_i(x) + S_{xi}^{+}\big(\sum_{j=1}^{n}\psi_{ij}t_j\big) + S_{xi}^{+}\Big\{\sum_{j=1}^{n}\psi_{ij}S_{xj}^{+}\big(\sum_{k=1}^{n}\psi_{jk}t_k\big)\Big\}, \qquad i = 1\ldots.n.\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
$$

If we denote by $f_i^{+}(x)$ the formal expansion to which the sequence $f_i^{(1)}(x)$, $f_i^{(2)}(x)$, $f_i^{(3)}(x)$, ... leads, we have

$$
\begin{aligned}
f_i^{+}(x) = t_i(x) &+ S_{xi}^{+}\big(\sum_{j=1}^{n}\psi_{ij}t_j\big) + S_{xi}^{+}\Big\{\sum_{j=1}^{n}\psi_{ij}S_{xj}^{+}\big(\sum_{k=1}^{n}\psi_{jk}t_k\big)\Big\}\\
&+ S_{xi}^{+}\Big[\sum_{j=1}^{n}\psi_{ij}S_{xj}^{+}\Big\{\sum_{k=1}^{n}\psi_{jk}S_{xk}^{+}\big(\sum_{l=1}^{n}\psi_{kl}t_l\big)\Big\}\Big] + \ldots., \qquad i = 1\ldots.n. \qquad (11)
\end{aligned}
$$

It is easy to verify that $f_i(x) = f_i^{+}(x)$, $i = 1\ldots.n$, formally satisfies (1). To this end let us perform on the n equations (11), in order, the operations denoted by D_i, $i = 1\ldots.n$, making use of the facts that $D_iS_{xi}^{+}(\eta) \equiv \eta$ and $D_it_i(x) \equiv 0$ and distributing the operations over the terms of the series in the second members; we have:

$$
\begin{aligned}
D_if_i^{+}(x) &= 0 + \sum_{j=1}^{n}\psi_{ij}t_j + \sum_{j=1}^{n}\psi_{ij}S_{xj}^{+}\big(\sum_{k=1}^{n}\psi_{jk}t_k\big) + \ldots.\\
&= \sum_{j=1}^{n}\psi_{ij}\Big\{t_j(x) + S_{xj}^{+}\big(\sum_{k=1}^{n}\psi_{jk}t_k\big) + \ldots.\Big\}\\
&= \sum_{j=1}^{n}\psi_{ij}(x)f_j^{+}(x), \qquad i = 1\ldots.n.
\end{aligned}
$$

We have thus shown that the functions (11) formally satisfy equations (1).

If now we start from the formal solution (10) instead of (9) and carry out an argument entirely analogous to the preceding one, we shall obtain another formal solution $f_i(x) = f_i^{-}(x)$, $i = 1\ldots.n$, representable as follows:

$$
\begin{aligned}
f_i^{-}(x) = t_i(x) &+ S_{xi}^{-}\big(\sum_{j=1}^{n}\psi_{ij}t_j\big) + S_{xi}^{-}\Big\{\sum_{j=1}^{n}\psi_{ij}S_{xj}^{-}\big(\sum_{k=1}^{n}\psi_{jk}t_k\big)\Big\}\\
&+ S_{xi}^{-}\Big[\sum_{j=1}^{n}\psi_{ij}S_{xj}^{-}\Big\{\sum_{k=1}^{n}\psi_{jk}S_{xk}^{-}\big(\sum_{l=1}^{n}\psi_{kl}t_l\big)\Big\}\Big] + \ldots., \qquad i = 1\ldots.n. \qquad (12)
\end{aligned}
$$

II.

In this section we determine the nature of the convergence and certain properties of the functions formally defined by the expansions already obtained. From the form of $\psi_{ij}(x)$ given in (2) it follows that there exists a constant M such that

$$|\psi_{ij}(x)| < \frac{M}{|x|^2}, \text{ for } |x| \geqq R. \qquad (13)$$

We assume that the functions $t_i(x)$, $i = 1 \ldots . n$, when not constant, are periodic functions of period 1 *analytic* throughout the finite plane. Let $x = u + v\sqrt{-1}$, u and v being real. We shall prove first that the expansions (11) are uniformly convergent series of analytic functions throughout a region in which

$$u \geqq R, \quad u \geqq Mn + 1 + \varepsilon, \quad \varepsilon = \text{any positive constant.} \tag{14}$$

Let us consider the series

$$\bar{S}^+_{xi} \left(\sum_{j=1}^{n} \psi_{ij}\, t_j\right) + \bar{S}^+_{xi} \left\{\sum_{j=1}^{n} \psi_{ij}\, \bar{S}^+_{xj} \left(\sum_{k=1}^{n} \psi_{jk}\, t_k\right)\right\} + \ldots, \tag{15}$$

where $\bar{S}^+_{xi}$ is defined by $\bar{S}^+_{xi}(\eta) \equiv |S^+_{xi}(|\eta|)|$. For fixed v and varying u let A_{vi} be the maximum value of $|t_i(x)|$. Let A_v be the greatest of the quantities A_{vi}, $i = 1 \ldots . n$. Using this notation and employing inequalities (13) and (14) it is clear that we may write

$$\bar{S}^+_{xi} \left(\sum_{j=1}^{n} \psi_{ij}\, t_j\right) < Mn\, A_v\, \bar{S}^+_{xi} \left(\frac{1}{x^2}\right). \tag{16}$$

Let us consider the case in which S^+_{xi} is an inverse of Δ. Then we have

$$\bar{S}^+_{xi} \left(\frac{1}{x^2}\right) = \frac{1}{|x|^2} + \frac{1}{|x+1|^2} + \frac{1}{|x+2|^2} + \ldots,$$

$$\therefore\ \bar{S}^+_{xi} \left(\frac{1}{x^2}\right) \leqq \frac{1}{u^2} + \frac{1}{(u+1)^2} + \frac{1}{(u+2)^2} + \ldots.$$

But the sum of the last series is less than the value of the integral

$$\int_{u-1}^{\infty} \frac{dz}{z^2} = \frac{1}{u-1}.$$

Hence

$$\bar{S}^+_{xi} \left(\sum_{j=1}^{n} \psi_{ij}\, t_j\right) < \frac{Mn\, A_v}{u-1}, \text{ when } S^+_{xi} \text{ is an inverse of } \Delta. \tag{17}$$

In the other case, namely that in which S^+_{xi} is an inverse of D, we have

$$\bar{S}^+_{xi} \left(\frac{1}{x^2}\right) \leqq \int_{u}^{\infty} \frac{M}{u^2}\, du = \frac{M}{u}.$$

Hence

$$\bar{S}^+_{xi} \left(\sum_{j=1}^{n} \psi_{ij}\, t_j\right) < \frac{Mn\, A_v}{u}, \text{ when } S^+_{xi} \text{ is an inverse of } D. \tag{18}$$

Combining (17) and (18) we have

$$\bar{S}^+_{xi} \left(\sum_{j=1}^{n} \psi_{ij}\, t_j\right) < \frac{Mn\, A_v}{u-1}, \qquad i = 1 \ldots . n;$$

and therefore

$$\left|S^+_{xi} \left(\sum_{j=1}^{n} \psi_{ij}\, t_j\right)\right| < \frac{Mn\, A_v}{u-1}, \qquad i = 1 \ldots . n.$$

Furthermore, since at any finite point x, for which relations (14) are satisfied, each of the n^2+n functions $\psi_{ij}(x)$, $t_j(x)$ is analytic, it is evident from the preceding argument that $S_{xi}^{+}(\psi_{ij}t_j)$ is analytic throughout the portion of the finite plane defined by (14).

Let us now consider the second term in (17). In view of the preceding discussion it is easy to obtain step by step the results indicated in the following relations:

$$\begin{aligned}\bar{S}_{xi}^{+}\{\sum_{j=1}^{n}\psi_{ij}\bar{S}_{xj}^{+}(\sum_{k=1}^{n}\psi_{jk}t_k)\} &\leqq Mn\,A_v\,\bar{S}_{xi}^{+}\Big\{\sum_{j=1}^{n}|\psi_{ij}|\,\bar{S}_{xj}^{+}\Big(\frac{1}{x^2}\Big)\Big\}\\ &< Mn\,A_v\,\bar{S}_{xi}^{+}\Big\{\sum_{j=1}^{n}|\psi_{ij}|\,\frac{1}{u-1}\Big\}\\ &< \frac{Mn\,A_v}{u-1}\bar{S}_{xi}^{+}\{\sum_{j=1}^{n}|\psi_{ij}|\}\\ &< \frac{M^2n^2A_v}{(u-1)^2}.\end{aligned}$$

One also proves readily that $S_{xi}^{+}\{\sum_{j=1}^{n}\psi_{ij}S_{xj}^{+}(\sum_{k=1}^{n}\psi_{jk}t_k)\}$ is an analytic function of x in the region defined by (14), and that its absolute value is less than $M^2n^2A_v/(u-1)^2$. By a continuation of the process it may be shown that every term of the series in (11) is analytic and that the series itself is term by term less in absolute value than

$$A_v+\frac{Mn}{u-1}A_v+\Big(\frac{Mn}{u-1}\Big)^2A_v+\Big(\frac{Mn}{u-1}\Big)^3A_v+\ldots\ldots \qquad (19)$$

But for u satisfying (14) this last series is uniformly convergent. The same is therefore true of the series in (11), and hence the latter converges to a function which is analytic throughout the portion of the finite plane defined by (14). It is easy to see then that (11) affords a solution of (1) in the region defined.

Starting from (15) and (16) and working in a similar manner, we may prove the convergence of the n series (11) in a different region of the plane. We suppose in this case that $|v|\geqq R$. We have

$$\bar{S}_{xi}^{+}\Big(\frac{1}{x^2}\Big)=\sum_{r=1}^{\infty}\frac{1}{(u+r)^2+v^2},$$

where S_{xi}^{+} is an inverse of Δ. But it is evident that

$$\sum_{r=1}^{\infty}\frac{1}{(u+r)^2+v^2}<\sum_{r=-\infty}^{\infty}\frac{1}{(u+r)^2+v^2}<2\Big\{\int_0^{\infty}\frac{dz}{z^2+v^2}+\frac{1}{v^2}\Big\}.$$

Hence

$$\bar{S}_{xi}^{+}\Big(\frac{1}{x^2}\Big)<\frac{\pi}{|v|}+\frac{2}{v^2}.$$

Comparing with (16) we have

$$\bar{S}_{xi}^{+}\,(\sum_{j=1}^{n}\psi_{ij}\,t_j) < Mn\,A_v\Big(\frac{\pi}{|v|}+\frac{2}{v^2}\Big), \qquad (20)$$

where S_{xi}^{+} is an inverse of Δ. Now when S_{xi}^{+} is an inverse of D it is easy to show that

$$\bar{S}_{xi}^{+}\Big(\frac{1}{x^2}\Big) < 2\int_0^{\infty}\frac{du}{u^2+v^2};$$

whence it follows that (20) holds also for these values of i. It is clear that one may continue the process just as in the preceding argument and thus show that each series (11) is term by term less in absolute value than the series

$$A_v + Mn\Big(\frac{\pi}{|v|}+\frac{2}{v^2}\Big)A_v + M^2n^2\Big(\frac{\pi}{|v|}+\frac{2}{v^2}\Big)^2A_v + \ldots\ldots \qquad (21)$$

Furthermore, for x in a region in which

$$|v| \geqq R,\quad Mn\Big(\frac{\pi}{|v|}+\frac{2}{v^2}\Big) \leqq 1-\varepsilon,\quad \varepsilon \text{ any positive constant}, \qquad (22)$$

it is easy to carry out an argument similar to that used in the preceding case, and to show that in such a region every term of the series in (11) is an analytic function, that the series as a whole converges uniformly (in any closed part of the region) and that therefore it represents a function of x which is analytic throughout the whole region.

It is obvious that the two regions thus found overlap, but that each one contains a part which is not in the other. The form of the total region, which is composed of these two, is easily made out from inequalities (14) and (22).

From the form of this region and the statements connected with (19) and (21) it follows that when x approaches infinity along any straight line parallel to the positive real axis we must have

$$\lim\,\{f_i^{+}(x) - t_i(x)\} = 0.$$

The principal results thus obtained are stated in the following theorem:

I. *The system of mixed equations*

$$D_i f_i(x) = \sum_{j=1}^{n}\psi_{ij}(x)f_j(x), \qquad i = 1\ldots.n,$$

where

$$\psi_{ij}(x) = \psi_{ij}''x^{-2} + \psi_{ij}'''x^{-3} + \ldots., \qquad |x| \geqq R,$$

has a solution

$$f_i(x) = f_i^{+}(x), \qquad i = 1\ldots.n,$$

where

$$f_i^{+}(x) = t_i(x) + S_{xi}^{+}(\sum_{j=1}^{n}\psi_{ij}t_j) + S_{xi}^{+}\{\sum_{j=1}^{n}\psi_{ij}S_{xj}^{+}(\sum_{k=1}^{n}\psi_{jk}t_k)\}$$
$$+ S_{xi}^{+}[\sum_{j=1}^{n}\psi_{ij}S_{xj}^{+}\{\sum_{k=1}^{n}\psi_{jk}S_{xk}^{+}(\sum_{l=1}^{n}\psi_{kl}t_l)\}] + \ldots., \qquad i = 1\ldots.n,$$

where $t_i(x)$ *is a constant or a periodic function of* x *of period* 1, *analytic throughout the finite plane, according as* $D_i \equiv D$ *or* $D_i \equiv \Delta$. *Each function of this solution is analytic throughout that portion of the finite plane for which either of the following sets of relations is true:*

$$u \geqq R, \quad u \geqq Mn + 1 + \varepsilon, \tag{23}$$

$$|v| \geqq R, \quad Mn\left(\frac{\pi}{|v|} + \frac{2}{v^2}\right) \leqq 1 - \varepsilon,$$

where M *is a properly chosen positive constant,* ε *is any positive constant whatever, and* u *and* v *are real, being defined by* $x = u + v\sqrt{-1}$. *Further, if* x *approaches infinity along any straight line parallel to the positive real axis then*

$$\lim \{f_i^+(x) - t_i(x)\} = 0. \tag{24}$$

It is clear that the formal expansions (12) may be treated in a manner entirely analogous to that just employed in the investigation of the properties of (11) and that corresponding results will be found which are in every respect parallel to those in Theorem I.

II. *With the following verbal alterations I is changed into the new theorem which is thus obtained: Replace* f^+ *by* f^-, S_{xi}^+ *by* S_{xi}^- *and instead of* (23) *write*

$$u \leqq -R, \quad u \leqq -(Mn + 1 + \varepsilon).$$

We have thus established the remarkable fact that there are two classes of solutions of (1) which are equally simple, the first of them being related to the right side of the plane as the second is to the left, and *vice versa.*

REMARK 1. It is clear that if the periodic functions $t_i(x), i = k + 1 \ldots n$, are not assumed to be analytic throughout the finite plane, the general convergence proof can be carried out in a manner analogous to the foregoing; that the solutions so obtained are analytic in the same region as before except for singularities at the singularities of $t_i(x), i = k + 1 \ldots n$, and that the limit for x approaching infinity along a line parallel to the positive real axis in general exists and has the same form as in the preceding cases.

REMARK 2. The preceding results are readily applied to a single mixed equation of order higher than the first. It will be sufficient to illustrate this remark by means of an example. For instance, if (1) has the following special form

$$g'(x) = \psi_{11}(x)g(x) + \psi_{12}(x)f(x),$$
$$\Delta f(x) = \psi_{21}(x)g(x) + \psi_{22}(x)f(x),$$

and if we differentiate the second of these equations with respect to x and

eliminate $g(x)$ and $g'(x)$ from the two given equations and the one thus obtained, we have

$$\Delta f'(x) - \psi_{22}(x) f'(x) - \left\{ \psi_{11}(x) + \frac{\psi'_{21}(x)}{\psi_{21}(x)} \right\} \Delta f(x)$$
$$+ \left\{ \psi_{22}(x) \frac{\psi'_{21}(x)}{\psi_{21}(x)} - \psi'_{22}(x) - \psi_{12}(x)\psi_{21}(x) + \psi_{11}(x)\psi_{22}(x) \right\} f(x) = 0,$$

an equation of the type studied by Poisson. Hence for any equation

$$\Delta f'(x) + af'(x) + b\Delta f(x) + cf(x) = 0,$$

which may be written in the preceding form, theorems I and II assert the existence of integrals having specified properties. Whether a given equation is reducible to this form is easily determined in any special case. Also, suitable values of the ψ's for effecting this reduction, in case it is possible, may be readily obtained.

III.

So far no essential use has been made of the fact that (1) is a *mixed* system; and therefore the results which we have obtained are true if every D_i is identical with D or if every D_i is identical with Δ. Hence our theorems are true of a system either of differential equations or of difference equations. If one seeks to apply them to differential equations it turns out that they are not suitable for deriving the existence theorem in a form of greatest simplicity; this is owing to the particular path of integration which has been chosen, there being in this case of course an infinity of paths from which choice can be made.

But with the difference equations the matter is essentially otherwise; the *path of summation* (to use a term analogous to *the path of integration*) for the sum which gives an inverse of Δ must have infinity for one of its extremities, must pass through the point x and be parallel to the axis of reals. This gives only two essentially different paths of summation; namely, the two extending from x to infinity along the two directions of this line. But, for the case in which (1) is a system of difference equations these two are entirely sufficient, as we shall see, to lead to the existence theorem in a form possessing the desired simplicity.*

* Compare the analogous existence theorem for a different system of equations treated in my thesis. See also a paper by G. D. Birkhoff, *Transactions of the American Mathematical Society*, Vol. XII (1911), pp. 243–284. The theorem here derived is a special case of Birkhoff's results. It may also be readily obtained as a corollary to the treatment in my thesis. It is nevertheless interesting to have a proof of it by the relatively simple method of the present paper, especially since it comes out as a corollary of the theorems in Section II.

If every D_i is identical with Δ, so that (1) is a system of difference equations, we may write this system in the form

$$f_i(x+1)=\sum_{j=1}^{n}\phi_{ij}(x)f_j(x), \quad \phi_{ij}(x)=\delta_{ij}+\psi_{ij}(x), \qquad i, j=1\ldots.n, \tag{25}$$

where δ_{ij} equals 0 or 1 according as i is or is not equal to j.

From Theorem II it follows that there exists a line l perpendicular to the negative axis of reals and such that the solutions referred to in Theorem II are all analytic throughout the part of the finite plane which lies to the left of l. If we know $f_i^-(x_0)$, $i=1\ldots.n$, then (25) gives $f_i^-(x_0+1)$, $i=1\ldots.n$; this known, $f_i^-(x_0+2)$, $i=1\ldots.n$, is found; and so on. Hence equation (25) may be used to extend the solution $f_i^-(x)$ of Theorem II across the plane to the right; it is evident that this process leads to the determination of $f_i^-(x)$, $i=1\ldots.n$, throughout the finite plane and that the singularities of these functions occur at the points congruent on the right to the singularities of $\psi_{ij}(x)$, $i, j=1\ldots.n$. (A point A is said to be *congruent on the right* to a point B if $A-B$ is a positive integer; it is *congruent on the left* if $A-B$ is a negative integer.)

If (25) is solved for $f_i(x)$, $i=1\ldots.n$, in terms of $f_i(x+1)$, $i=1\ldots.n$, and x is replaced by $x-1$, the system may be written in the form

$$f_i(x-1)=\sum_{j=1}^{n}\bar{\psi}_{ij}(x)f_j(x), \qquad i=1\ldots.n.$$

If one starts from this equation and employs Theorem I, it is easy to show that the functions $f_i^+(x)$, $i=1\ldots.n$, are analytic throughout the finite plane except at points congruent on the left to the singularities of $\bar{\psi}_{ij}(x)$, $i, j=1\ldots.n$.

For the present case, namely that in which (1) is a system of difference equations, the periodic functions $t_i(x)$, $i=1\ldots.n$, may be removed from under the signs S_{xi}^+, $i=1\ldots.n$; that is,

$$S_{xi}^+(\eta t_j)=t_j S_{xi}^+(\eta).$$

From this it follows that $f_i^+(x)$ may be written in the form:

$$\begin{aligned} f_i^+(x)=\sum_{j=1}^{n}t_j(x)\Big[\delta_{ij}&+S_{xi}^+(\psi_{ij})+S_{xi}^+\{\sum_{\alpha=1}^{n}\psi_{i\alpha}S_{x\alpha}^+(\psi_{\alpha j})\} \\ &+S_{xi}^+[\sum_{\alpha=1}^{n}\psi_{i\alpha}S_{x\alpha}^+\{\sum_{\beta=1}^{n}\psi_{\alpha\beta}S_{x\beta}^+(\psi_{\beta j})\}]+\ldots.\Big], \end{aligned} \tag{26}$$

$$i=1\ldots.n; \quad \delta_{ij}=0 \text{ if } i\neq j; \quad \delta_{ij}=1 \text{ if } i=j.$$

As particular solutions of (1) we may therefore choose

$$f_{1j}^{+}(x),\ f_{2j}^{+}(x),\ \ldots,\ f_{nj}^{+}(x), \qquad j=1\ldots n,$$

where

$$f_{ij}^{+}(x)=\delta_{ij}+S_{xi}^{+}(\psi_{ij})+S_{xi}^{+}\{\sum_{\alpha=1}^{n}\psi_{i\alpha}S_{x\alpha}^{+}(\psi_{\alpha j})\}$$
$$+S_{xi}^{+}[\sum_{\alpha=1}^{n}\psi_{i\alpha}S_{x\alpha}^{+}\{\sum_{\beta=1}^{n}\psi_{\alpha\beta}S_{x\beta}^{+}(\psi_{\beta j})\}]+\ldots, \qquad i,j=1\ldots n. \quad (27)$$

Thus $f_{ij}^{+}(x)$, j fixed and $i=1\ldots n$, is a special case of the preceding solution $f_i^{+}(x)$.

Moreover, $f_{ij}^{+}(x)$, $i, j=1\ldots n$, constitute a fundamental system of solutions of (1); for they satisfy the necessary and sufficient condition of independence, namely that the determinant $|f_{ij}^{+}|\not\equiv 0$. To prove this we have only to notice that as x approaches infinity along a line parallel to the positive real axis $f_{ij}^{+}(x)$ approaches δ_{ij}; and hence, that the determinant $|f_{ij}^{+}|$ approaches 1. From the independence of these solutions it follows that the general solution of the system of difference equations is

$$f_i(x)=\sum_{j=1}^{n}\omega_j(x)f_{ij}^{+}(x), \qquad i=1\ldots n,$$

where $\omega_j(x)$, $j=1\ldots n$, are arbitrary periodic functions of x of period 1.

In a similar manner one arrives at a second fundamental system of solutions

$$f_{1j}^{-}(x),\ f_{2j}^{-}(x),\ \ldots,\ f_{nj}^{-}(x), \qquad j=1\ldots n,$$

where

$$f_{ij}^{-}(x)=\delta_{ij}+S_{xi}^{-}(\psi_{ij})+S_{xi}^{-}\{\sum_{\alpha=1}^{n}\psi_{i\alpha}S_{x\alpha}^{-}(\psi_{\alpha j})\}+\ldots, \qquad i,j=1\ldots n, \quad (28)$$

$f_{ij}^{-}(x)$, j fixed and $i=1\ldots n$, being a special case of $f_i^{-}(x)$, $i=1\ldots n$.

In case we have a solution which belongs to either of these fundamental solutions, the quantity A_v entering into (19) and (21) is equal to 1. Hence, if x approaches infinity along any ray from the origin, the negative real axis excepted, we have

$$\lim f_{ij}^{+}(x)=\delta_{ij},$$

with a like result for functions of the other fundamental system.

The principal results of this section may be constructed into the following theorem:

The system of difference equations

$$f_i(x+1)=\sum_{j=1}^{n}\phi_{ij}(x)f_j(x) \quad or \quad f_i(x-1)=\sum_{j=1}^{n}\bar{\psi}_{ij}(x)f_j(x), \qquad i=1\ldots n,$$

where

$$\phi_{ij}(x)=\delta_{ij}+\psi_{ij}(x),\ \psi_{ij}(x)=\psi_{ij}''x^{-2}+\psi_{ij}'''x^{-3}+\ldots, \qquad |x|\geqq R,$$

has two equally simple fundamental systems of solutions $f_{ij}^{+}(x)$, $f_{ij}^{-}(x)$, $i, j = 1 \ldots n$, *defined by means of equations* (27) *and* (28) *respectively; each function of the first solution is analytic throughout the finite plane except at points congruent on the right to the singularities of* $\bar{\psi}_{ij}(x)$, $i, j = 1 \ldots n$, *and each function of the second is analytic throughout the finite plane except at points congruent on the left to singularities of* $\psi_{ij}(x)$, $i, j = 1 \ldots n$.

If x approaches infinity along any ray from the origin, the negative real axis excepted, then

$$\lim f_{ij}^{+}(x) = \delta_{ij};$$

if along any ray except the positive real axis, then

$$\lim f_{ij}^{-}(x) = \delta_{ij}.$$

IV.

In the preceding section we have seen that when (1) is a system of difference equations the general solution (11) may be written in the form (26); and that this form has the property of being linear in the arbitrary periodic functions $t_i(x)$, $i = 1 \ldots n$. It is on account of the possibility of writing (11) in this form that we have been able to express the general solution of (25) in terms of a fundamental system consisting of n properly chosen particular solutions.

A reference to the discussion of this matter in Section III will bring out the fact that it depends essentially on the equation

$$S_{xi}^{+}(\eta t_j) = t_j S_{xi}^{+}(\eta),$$

where S_{xi}^{+} is an inverse of Δ. But when S_{xi}^{+} is a definite integral and $t_j(x)$ is a periodic function (not a constant) the corresponding equality does not exist. It follows, therefore, that the method which was effective in the case of difference equations for obtaining a fundamental system of solutions will not apply to the case of a mixed system of equations, and, in fact, simple examples are sufficient to indicate that the relations among the infinity of solutions of such systems are probably not expressible in a simple way in terms of a finite number of them singled out as fundamental. In connection with this remark one may compare the paper by Lecornu, to which reference has already been made.

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